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# Rings of integer-valued rational functions

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**Abstract**

Let $D$ be an integral domain which differs from its quotient field $K$. The ring of integer-valued rational functions of $D$ on a subset $E$ of $D$ is defined as $\mathrm{Int}^{\mathrm{R}}(E,D) = \{f(X) \in K(X) | f(E) \subseteq D\}$. We write $\mathrm{Int}^{\mathrm{R}}(D)$ for $\mathrm{Int}^{\mathrm{R}}(D,D)$.

It is easy to see that $\mathrm{Int}^{\mathrm{R}}(D)$ is strictly larger than the more familiar ring $\mathrm{Int}(D)$ of integer-valued polynomials precisely when there exists a polynomial $f(X) \in D[X]$ such that $f(d)$ is a unit in $D$ for each $d \in D$. In fact, there are striking differences between $\mathrm{Int}^{\mathrm{R}}(D)$ and $\mathrm{Int}(D)$ in many of the cases where they are not equal.

Rings of integer-valued rational functions have been studied in at least two previous papers. The purpose of this note is to consolidate and greatly expand the results of these papers. Among the topics included, we give conditions so that $\mathrm{Int}^{\mathrm{R}}(E,D)$ is a Prüfer domain, we study the value ideals of $\mathrm{Int}^{\mathrm{R}}(E,D)$ (for example, we show that $\mathrm{Int}^{\mathrm{R}}(K,D)$ satisfies the strong Skolem property provided it is a Prüfer domain), and we study the prime ideals of $\mathrm{Int}^{\mathrm{R}}(E,D)$ (for example, we show that if $V$ is a valuation domain, then each prime ideal of $\mathrm{Int}^{\mathrm{R}}(V)$ above the maximal ideal $m$ of $V$ is maximal if and only if $m$ is principal). 

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## 1. Introduction

Throughout this paper, $D$ denotes an integral domain which is not a field, with quotient field $K$, and $E$ a subset of $K$. The ring of integer-valued rational functions of

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$D$ on the subset $E$ is defined as the ring

$$\mathrm{Int}^{\mathrm{R}}(E,D) = \{f(X) \in K(X) \mid f(E) \subseteq D\}.$$

We simply write $\mathrm{Int}^{\mathrm{R}}(D)$ for $\mathrm{Int}^{\mathrm{R}}(D,D)$. Rings of integer-valued rational functions have been studied in at least two different papers, [3], and [11]. The purpose of this note is to consolidate and greatly expand the results of these papers.

The ring $\mathrm{Int}^{\mathrm{R}}(D)$ has many similarities to the familiar ring of integer-valued polynomials $\mathrm{Int}(D) = \{f(X) \in K[X] \mid f(D) \subseteq D\}$. In fact, for many familiar domains $D$ (such as $\mathbb{Z}$ in particular) $\mathrm{Int}(D) = \mathrm{Int}^{\mathrm{R}}(D)$. However, we shall observe some striking differences in many instances where they are not equal (as is always the case if $D$ is quasi-local). For example, if $V$ is a valuation ring with a maximal ideal which is not principal or with an infinite residue field, then $\mathrm{Int}(V) = V[X]$ and integer-valued polynomials are not much worth considering, whereas, if $V$ is a rank-one discrete valuation domain with finite residue field, $\mathrm{Int}(V)$ turns out to be a Prüfer domain [7, Proposition 2.3]. Also, if we looked at integer-valued polynomials on a subset, we should restrict ourselves to fractional subsets of $V$ (that is, subsets with a common denominator), since otherwise $\mathrm{Int}(E,V)$ contains only constants [13]. Integer-valued rational functions turn out to be much more interesting: we show that $\mathrm{Int}^{\mathrm{R}}(E,V)$ is not trivial and is a Prüfer domain, even a Bézout domain, whenever the maximal ideal $\mathfrak{m}$ of $V$ is principal or the residue field $V/\mathfrak{m}$ is not algebraically closed, whatever the subset $E$ of $K$ (and even for $E = K$). We also generalize such results to $\mathrm{Int}^{\mathrm{R}}(E,D)$, where $D$ is a Prüfer domain satisfying various hypotheses.

In Section 2 the principal results concern localization properties. In particular, we relate $\mathrm{Int}^{\mathrm{R}}(E,D)$ with $\mathrm{Int}^{\mathrm{R}}(K,D)$ and $\mathrm{Int}^{\mathrm{R}}(K,S^{-1}D)$, where $S^{-1}D$ is a localization of $D$.

In Section 3 we consider the question of characterizing the domains $D$ for which $\mathrm{Int}^{\mathrm{R}}(D)$ is a Prüfer (or Bézout) domain. An easy necessary condition is that $D$ itself be a Prüfer domain. We describe two classes of Prüfer domains such that $\mathrm{Int}^{\mathrm{R}}(E,D)$ is a Prüfer domain: *monic Prüfer domains* (such that there exists a monic unit-valued polynomial $f(X) \in D[X]$) and *singular Prüfer domains* (the definition of which is more technical). In particular, for a valuation domain $V$, these classes correspond to the cases where the maximal ideal of $V$ is principal and where its residue field is not algebraically closed.

In Section 4 we consider ideals of values of $\mathrm{Int}^{\mathrm{R}}(E,D)$. The major result is that $\mathrm{Int}^{\mathrm{R}}(K,D)$ satisfies the strong Skolem property whenever it is a Prüfer domain. We also discuss subsets $E$ of $K$ such that $\mathrm{Int}^{\mathrm{R}}(E,D)$ satisfies the strong Skolem property.

In Sections 5 and 6 we consider the prime ideal spectrum of $\mathrm{Int}^{\mathrm{R}}(E,D)$. First with no hypothesis on $D$, we consider the prime ideals above $(0)$ and prove in particular that there always exist nonzero such primes. We next show how to describe some prime ideals of $\mathrm{Int}^{\mathrm{R}}(E,D)$, using ultrafilters. In the second of these two sections, we let $D = V$ be a valuation domain (such that its maximal ideal is principal or its residue field is not algebraically closed) and consider the prime ideals of $\mathrm{Int}^{\mathrm{R}}(V)$ above the maximal ideal $\mathfrak{m}$ of $V$. One of our major results is that each prime ideal of $\mathrm{Int}^{\mathrm{R}}(V)$ above $\mathfrak{m}$ is maximal if and only if $\mathfrak{m}$ is principal.

## 2. Localization

The domains such that $\mathrm{Int}(D) = \mathrm{Int}^{\mathrm{R}}(D)$ have been studied in [2] and [9] where they were called d-rings. We are mainly interested in *non-d-rings*, for which equality does not occur. A necessary and sufficient condition for a domain to be a non-d-ring, is that there exists a nonconstant polynomial $f(X) \in D[X]$ such that $f(d)$ is a unit in $D$ for all $d \in D$ [9, Proposition 1]. Indeed this condition is clearly sufficient since then $1/f(X)$ lies in $\mathrm{Int}^{\mathrm{R}}(D)$, but is not a polynomial (note that if $J(D)$, the Jacobson radical of $D$, is nonzero and if $d$ is a nonzero element of $J(D)$, then $f(X) = dX + 1$ is such a unit-valued polynomial). But now suppose that $D = V$ is the ring of a valuation $v$, and that $f(X)$ is a monic unit-valued polynomial of $V[X]$ (that is, taking unit values on $V$), then $1/f(X)$ lies in $\mathrm{Int}^{\mathrm{R}}(K, V)$ and so does $X/f(X)$: if $f(X)$ is of degree $n$, and if $v(x) < 0$, then $v(f(x)) = nv(x) < v(x)$ (necessarily $n \geq 2$). Along that line we have the following:

**Proposition 2.1.** *Let $D$ be a domain and $U(D)$ be the set of units of $D$. Suppose there exists a rational function $\psi$ such that*
- *$\psi$ and $X\psi$ both belong to* $\mathrm{Int}^{\mathrm{R}}(K,D)$,
- *$\psi(D) \subseteq U(D)$.*

*Then $K(X)$ is the quotient field of* $\mathrm{Int}^{\mathrm{R}}(K,D)$ *and* $\mathrm{Int}^{\mathrm{R}}(E,D)$ *is the localization of* $\mathrm{Int}^{\mathrm{R}}(K,D)$ *with respect to the multiplicative set*

$$S = \{\rho \in \mathrm{Int}^{\mathrm{R}}(K,D) \mid \rho(E) \subseteq U(D)\}.$$

**Proof.** If $\phi \in K(X)$, then $\psi(\phi)$ and $\phi\psi(\phi)$ both belong to $\mathrm{Int}^{\mathrm{R}}(K,D)$. Since $\phi = (\psi(\phi))^{-1}(\phi\psi(\phi))$, it follows that $K(X)$ is the quotient field of $\mathrm{Int}^{\mathrm{R}}(K,D)$. Now suppose that $\phi \in \mathrm{Int}^{\mathrm{R}}(E,D)$. By hypothesis $\psi(\phi)$ is unit-valued on $E$, hence $\psi(\phi) \in S$. □

We now give examples where $D$ is a Prüfer domain. For each maximal ideal $\mathfrak{m}$ of $D$ we denote by $v_{\mathfrak{m}}$ the corresponding valuation. If $\psi = 1/f(X)$, where $f(X) \in D[X]$, belongs to $\mathrm{Int}^{\mathrm{R}}(K,D)$, note that $f$ must necessarily be a unit-valued polynomial (hence the last hypothesis becomes superfluous in the previous proposition). Note also that $f$ is unit-valued on $D$ if and only if, for each maximal ideal $\mathfrak{m}$ of $D$, it is unit-valued on $D_{\mathfrak{m}}$ (that is, $f$ has no root modulo $\mathfrak{m}$). Lastly, note that such a unit-valued polynomial is such that both $1/f(X)$ and $X/f(X)$ belong to $\mathrm{Int}^{\mathrm{R}}(K,D)$ if and only if, there is a family $\Lambda$ of maximal ideals of $D$ such that $D = \bigcap_{\mathfrak{m}\in\Lambda} D_{\mathfrak{m}}$ and, for each maximal ideal $\mathfrak{m} \in \Lambda$, if $x \in K$ is such that $v_{\mathfrak{m}}(x) < 0$, then $v_{\mathfrak{m}}(f(x)) \leq v_{\mathfrak{m}}(x) < 0$.

**Example 2.2.** (1) Suppose that $D$ is a Prüfer domain such that there exists a monic unit-valued polynomial $f(X) \in D[X]$. Then $1/f(X)$ and $X/f(X)$ belong to $\mathrm{Int}^{\mathrm{R}}(K,D)$. We shall say that $D$ is *monic*. In particular, a valuation domain is monic if and only if its residue field is not algebraically closed.

(2) Suppose that $D$ is a Prüfer domain and that there exists a family $\Lambda$ of maximal ideals of $D$ such that

- $D = \bigcap_{\mathfrak{m}\in\Lambda} D_{\mathfrak{m}}$,
- for each $\mathfrak{m} \in \Lambda$, $\mathfrak{m}D_{\mathfrak{m}} = t_{\mathfrak{m}}D_{\mathfrak{m}}$ is a principal ideal,
- there is an element $t \in D$ and an integer $n$ such that, for each $\mathfrak{m} \in \Lambda$, $0 < v_{\mathfrak{m}}(t) < nv_{\mathfrak{m}}(t_{\mathfrak{m}})$.

If $f = 1 + tX^n$, then $1/f(X)$ and $X/f(X)$ belong to $\mathrm{Int}^{\mathrm{R}}(K,D)$. We shall say that $D$ is *singular*. In particular, a valuation domain is singular if and only if its maximal ideal $\mathfrak{m}$ is principal: $\mathfrak{m} = tD$; in this case we may simply consider the polynomial $f(X) = 1 + tX^2$.

Note that the monic case is realized if the residue fields of $D$ are all finite with a bound on their order and that the singular case is realized if $D$ is a Dedekind domain, all its essential valuations being an extension of a rank-one discrete valuation, with a bound on their ramification index.

Under similar hypotheses, we next state another localization property:

**Proposition 2.3.** *Let $D$ be a domain. Suppose there exists a polynomial $f(X) \in D[X]$ such that $1/f(X)$ and $X/f(X)$ both belong to $\mathrm{Int}^{\mathrm{R}}(K,D)$. Let $S^{-1}D$ be a localization of $D$ and $U(S^{-1}D)$ be the set of units of $S^{-1}D$. Then $\mathrm{Int}^{\mathrm{R}}(E,S^{-1}D)$ is the localization of $\mathrm{Int}^{\mathrm{R}}(E,D)$, with respect to the multiplicative set*

$$T = \{\phi \in \mathrm{Int}^{\mathrm{R}}(E,D) \mid \phi(E) \subseteq U(S^{-1}D)\}.$$

**Proof.** It is clear that $T^{-1}\,\mathrm{Int}^{\mathrm{R}}(E,D) \subseteq \mathrm{Int}^{\mathrm{R}}(E,S^{-1}D)$. Let $f(X) \in D[X]$ be as in the statement of the proposition and let $\phi \in \mathrm{Int}^{\mathrm{R}}(E,S^{-1}D)$. As in the previous proof, write $\phi = (1/f(\phi))^{-1}\,\phi/f(\phi)$. We need to show that $1/f(\phi)$ is a unit in $\mathrm{Int}^{\mathrm{R}}(E,S^{-1}D)$. Equivalently, it suffices to show that $f(\phi(d)) \in U(S^{-1}D)$ for every $d \in E$. Let $d \in E$. Suppose $\phi(d) \in D$. Necessarily, $f(\phi(d))$ is a unit in $D$ and so is a unit in $S^{-1}D$ as well. Now, suppose that $\phi(d) \notin D$. We know that $\phi(d) \in S^{-1}D$ and that $f(X) \in D[X]$. It follows that $f(\phi(d)) \in S^{-1}D$. Since $1/f(\phi(d)) \in D \subseteq S^{-1}D$ the result follows. □

To conclude this section, we show that, if the quotient field of a valuation domain $V$ is algebraically closed, it does not satisfy the hypotheses of Proposition 2.1 (since it does not satisfy its conclusions). Incidentally, this is a way to see that, in this case, the residue field of $V$ is algebraically closed and its maximal ideal is not principal.

**Proposition 2.4.** *Let $V$ be a valuation domain with maximal ideal $\mathfrak{m}$ and quotient field $K$. Suppose that $K$ is algebraically closed. Then $\mathrm{Int}^{\mathrm{R}}(K,V) = V$ and $\mathrm{Int}^{\mathrm{R}}(V) = S^{-1}V[X]$ where $S = \{rX + 1 \mid r \in \mathfrak{m}\}$.*

**Proof.** Let $E$ be a subset of $K$ and $\phi \in \mathrm{Int}^{\mathrm{R}}(E,D)$ be a nonzero rational function. We can write $\phi = h/g$ where $h$ and $g$ both lie in $V[X]$ and are relatively prime over $K[X]$. If $d \in E$ is a root of $g$, then $\phi(d)$ is undefined. Hence, $g$ cannot have any roots in $E$. If $K$ is algebraically closed, and if $E = K$, then $g$ must be constant, hence $\mathrm{Int}^{\mathrm{R}}(K, V) = V$. With the same hypothesis, if $E = V$, then $g$ can be factored into linear factors over $K$, of the form $rX + 1$, where $r \in \mathfrak{m}$ (since $g$ has no root in $V$). The result follows. □

## 3. Prüfer/Bézout domains

In this section, we consider the question of classifying the domains $D$ such that $\mathrm{Int}^{\mathrm{R}}(E, D)$ is a Prüfer (or Bézout) domain for various subsets $E$ of $K$. We begin with a necessary condition which generalizes [11, Proposition 3.11].

**Proposition 3.1.** *Let $D$ be a domain such that $\mathrm{Int}^{\mathrm{R}}(E, D)$ is a Prüfer domain. Then $D$ is a Prüfer domain.*

Indeed, $D$ is an homomorphic image of $\mathrm{Int}^{\mathrm{R}}(E, D)$ (choose $a \in E$ and consider the morphism $\phi \in \mathrm{Int}^{\mathrm{R}}(E, D) \to \phi(a)$).

We shall now prove that $\mathrm{Int}^{\mathrm{R}}(K, D)$ is a Prüfer domain in both the monic and singular case of Examples 2.2 (and even a Bézout domain in the latter case). The next result generalizes [11, Theorem 3.3]; its proof is in every respect similar (replacing $\mathrm{Int}^{\mathrm{R}}(D)$ by $\mathrm{Int}^{\mathrm{R}}(E, D)$).

**Theorem 3.2.** *Let $D$ be a monic Prüfer domain. Then $\mathrm{Int}^{\mathrm{R}}(K, D)$ is a Prüfer domain and its Picard group is torsion. More precisely, suppose that there is a degree $n$ monic unit-valued polynomial, with coefficients in $D$. Then, for each finitely generated ideal $\mathfrak{A}$ of $\mathrm{Int}^{\mathrm{R}}(K, D)$, there is an integer $s$ such that $\mathfrak{A}^{n^s}$ is principal.*

The following corollary follows immediately.

**Corollary 3.3.** *Let $D$ be a monic Prüfer domain. Suppose there exist two monic unit-valued polynomials $f_1$ and $f_2 \in D[X]$ the degrees of which are relatively prime. Then $\mathrm{Int}^{\mathrm{R}}(E, D)$ is a Bézout domain.*

Note that the hypotheses of this corollary are easily satisfied: for example, choose two monic irreducible polynomials $f$ and $g$ in $\mathbb{Z}[X]$ with relatively prime degrees (each degree greater than 1) and let $D = \mathbb{Z}[\{f(d)^{-1}, g(d)^{-1} \mid d \in \mathbb{Z}\}]$. Then, $D$ is a Dedekind domain where $f$ and $g$ are both unit-valued on $D$ (see [10, Proposition 1.14 and Construction 1.16] for details).

We shall next deal with singular Prüfer domains and first give a sufficient condition for $\mathrm{Int}^{\mathrm{R}}(K, D)$ to be a Bézout domain (and thus also $\mathrm{Int}^{\mathrm{R}}(E, D)$ for each subset $E$ of $K$).

**Lemma 3.4.** *Let $D$ be a Prüfer domain. Suppose there exists a family $\Lambda$ of maximal ideals such that $D = \bigcap_{\mathfrak{m} \in \Lambda} D_{\mathfrak{m}}$ and a rational function $\theta \in K(X)$ such that, for each*

*$\mathfrak{m} \in \Lambda$ and each $x \in K$,*
*– if $v_\mathfrak{m}(x) \neq 0$, then $v_\mathfrak{m}(\theta(x)) = 0$,*
*– if $v_\mathfrak{m}(x) = 0$, then $v_\mathfrak{m}(\theta(x)) > 0$.*
*Then* $\mathrm{Int}^\mathrm{R}(K,D)$ *is a Bézout domain.*

**Proof.** Let $\phi$, $\psi \in \mathrm{Int}^\mathrm{R}(K,D)$ and let $\mathfrak{A} = (\phi, \psi)$. We want to show that $\mathfrak{A}$ is principal. Clearly, $\theta \in \mathrm{Int}^\mathrm{R}(K,D)$. Hence, $\rho = \theta(\phi/\psi)\phi + \psi$ belongs to $\mathfrak{A}$. It is easy to see that, for all $\mathfrak{m} \in \Lambda$ and all $x \in K$,

$$v_\mathfrak{m}(\rho(x)) = \inf\{v_\mathfrak{m}(\phi(x)), v_\mathfrak{m}(\psi(x))\}.$$

It immediately follows that $\mathfrak{A} = \rho\,\mathrm{Int}^\mathrm{R}(K,D)$. □

We next show that a singular Prüfer domain is such that $\mathrm{Int}^\mathrm{R}(K,D)$ is a Bézout domain, thus, generalizing [3, Corollaire 7.4] (which is concerned with a rank-one discrete valuation domain). From Proposition 2.1, $\mathrm{Int}^\mathrm{R}(E,D)$ is then also a Bézout domain for each subset $E$ of $K$.

**Theorem 3.5.** *Let $D$ be a singular Prüfer domain. Then* $\mathrm{Int}^\mathrm{R}(K,D)$ *is a Bézout domain.*

**Proof.** Recall, from the definition, that if $D$ is a singular Prüfer domain, there exists a family $\Lambda$ of maximal ideals of $D$ such that
- $D = \bigcap_{\mathfrak{m}\in\Lambda} D_\mathfrak{m}$,
- for each $\mathfrak{m} \in \Lambda$, $\mathfrak{m}D_\mathfrak{m} = t_\mathfrak{m}D_\mathfrak{m}$ is a principal ideal,
- there is an element $t \in D$ and an integer $n$ such that, for each $\mathfrak{m} \in \Lambda$, $0 < v_\mathfrak{m}(t) < n v_\mathfrak{m}(t_\mathfrak{m})$.

One may then verify that the function

$$\theta = \frac{t(1+X^{2n})}{(1+tX^n)(t+X^n)},$$

satisfies the hypothesis of the previous lemma. □

**Remark 3.6.** (i) Let $V$ be a valuation domain with a finite residue field. It results from Corollary 3.3 that $\mathrm{Int}^\mathrm{R}(E,V)$ is a Bézout domain. In this case, it would also be easy to give a function satisfying the hypotheses of Lemma 3.4:
– If the cardinal of the residue field is $q \neq 2$, we could let

$$\theta = \frac{1-X^{q-1}}{1+X-X^{q-1}}.$$

– If the cardinal of the residue field is $q = 2$, we could let

$$\theta = \frac{1+X^2}{1+X+X^2}.$$

(ii) If $D$ is a valuation domain, it is singular if and only if its maximal ideal is principal. If $t$ is a generator, the function

$$\rho = \frac{t(1+X^2)}{(1+tX^2)}$$

is such that
– if $v(x) < 0$, then $v(\rho(x)) = 0$,
– if $v(x) \geq 0$, then $v(\rho(x)) > 0$.

Hence $\theta(X) = \rho(X) + \rho(1/X)$ satisfies the hypothesis of Lemma 3.4.

(iii) It is clear that the previous theorem does apply, with the same proof, to the ring of integer-valued rational functions, in several variables (taking $K^n$ into $D$). Therefore, this ring of integer-valued rational functions in several variables is a Bézout domain.

(iv) The results of this section show that in many cases in which $\mathrm{Int}^{\mathrm{R}}(E,D)$ is a Prüfer domain, it is actually a Bézout domain. We may ask whether this is always the case. In particular, if $D$ is a valuation domain with residue field equal to the field $\mathbb{R}$ of real numbers and with a nonprincipal maximal ideal.

(v) According to Proposition 2.4, if $V$ is a valuation domain and its quotient field is algebraically closed, then $\mathrm{Int}^{\mathrm{R}}(V) = S^{-1}V[X]$, where $S = \{rX+1 \mid r \in \mathfrak{m}\}$. In this case, $\mathrm{Int}^{\mathrm{R}}(V)$ is not a Prüfer domain, indeed the maximal ideal $(\mathfrak{m}, X)$ of $V[X]$ survives in this localization and it contains incomparable height-one primes (such as the principal prime ideals generated by polynomials of the form $X-d$ where $d \in \mathfrak{m}$).

## 4. Skolem properties

Let $\mathfrak{A}$ be an ideal of $\mathrm{Int}^{\mathrm{R}}(E,D)$ and let $x \in E$. Then $\mathfrak{A}(x) = \{\phi(x) \mid \phi \in \mathfrak{A}\}$ is an ideal of $D$, called the value ideal of $\mathfrak{A}$ at $x$. The Skolem properties deal with the question of the extent to which a finitely generated ideal is characterized by its value ideals. We say that

- $\mathrm{Int}^{\mathrm{R}}(E,D)$ satisfies the Skolem property provided whenever $\mathfrak{A}$ is a finitely generated ideal of $\mathrm{Int}^{\mathrm{R}}(E,D)$ such that $\mathfrak{A}(x) = D$ for all $x \in E$, then $\mathfrak{A} = \mathrm{Int}^{\mathrm{R}}(E,D)$.
- $\mathrm{Int}^{\mathrm{R}}(E,D)$ satisfies the strong Skolem property provided whenever $\mathfrak{A}$ and $\mathfrak{B}$ are finitely generated ideals of $\mathrm{Int}^{\mathrm{R}}(E,D)$ such that $\mathfrak{A}(x) = \mathfrak{B}(x)$ for all $x \in E$, then $\mathfrak{A} = \mathfrak{B}$.

We first deal with the Skolem property, with no restriction on $E$.

**Lemma 4.1.** *Let $D$ be a Prüfer domain and let $E$ be a subset of $K$ such that $\mathrm{Int}^{\mathrm{R}}(E,D)$ is a Prüfer domain. Then $\mathrm{Int}^{\mathrm{R}}(E,D)$ satisfies the Skolem property.*

**Proof.** For each $x \in E$ and each prime ideal $\mathfrak{p}$ of $D$,

$$V_{\mathfrak{p},x} = \{f(X) \in K(X) | f(x) \in D_{\mathfrak{p}}\}$$

is a valuation overring of $\mathrm{Int}^{\mathrm{R}}(E,D)$ and $\mathrm{Int}^{\mathrm{R}}(E,D)$ is the intersection of these valuation overrings, where $x$ runs over the elements of $E$ and $\mathfrak{p}$ over the prime ideals of $D$ (we may also restrict ourselves to the maximal ideals). The intersection of the maximal ideal of $V_{\mathfrak{p},x}$ with $\mathrm{Int}^{\mathrm{R}}(E,D)$ is the prime ideal

$$\mathfrak{M}_{\mathfrak{p},x} = \{f(X) \in \mathrm{Int}^{\mathrm{R}}(E,D) | f(x) \in \mathfrak{p}\},$$

and $V_{\mathfrak{p},x}$ is the localization of $\mathrm{Int}^{\mathrm{R}}(E,D)$ with respect to this prime ideal (since $\mathrm{Int}^{\mathrm{R}}(E,D)$ is a Prüfer domain). From [8, Corollary 1.5], a Prüfer domain $R$ is the intersection of the localizations of $R$ at some prime ideals $\mathfrak{P}_i$ if and only if each proper finitely generated ideal is contained in one of these prime ideals. Therefore, if a finitely generated ideal $\mathfrak{A}$ of $\mathrm{Int}^{\mathrm{R}}(E,D)$ is a proper ideal, it is contained in one of the prime ideals $\mathfrak{M}_{\mathfrak{p},x}$, which implies that the value ideal $\mathfrak{A}(x)$ is proper (it is contained in the prime ideal $\mathfrak{p}$). This is the Skolem property. □

In fact, if $\mathrm{Int}^{\mathrm{R}}(E,D)$ is a Prüfer domain, it satisfies the strong Skolem property, provided that $E$ is large enough (a sufficient condition, according to our next result, is that it contains $D$. Thus, in particular, $\mathrm{Int}^{\mathrm{R}}(D)$ and $\mathrm{Int}^{\mathrm{R}}(K,D)$ satisfy the strong Skolem property).

**Theorem 4.2.** *Let $D$ be a Prüfer domain and let $E$ be a subset of $K$ such that $\mathrm{Int}^{\mathrm{R}}(E,D)$ is a Prüfer domain. Suppose that, for each maximal ideal $\mathfrak{m}$ of $D$, $E$ contains elements with arbitrarily large values for the corresponding valuation $v_{\mathfrak{m}}$. Then $\mathrm{Int}^{\mathrm{R}}(E,D)$ satisfies the strong Skolem property.*

**Proof.** Let $\mathfrak{A}$ and $\mathfrak{B}$ be two finitely generated ideals such that $\mathfrak{A}(x) = \mathfrak{B}(x)$, for all $x \in E$. Without loss of generality, we may assume that $\mathfrak{A} \subseteq \mathfrak{B}$. Since $\mathrm{Int}^{\mathrm{R}}(E,D)$ is a Prüfer domain, $\mathfrak{B}$ is invertible and $\mathfrak{J} = \mathfrak{A}\mathfrak{B}^{-1}$ is an integral ideal of $\mathrm{Int}^{\mathrm{R}}(E,D)$. Since $\mathfrak{B}\mathfrak{B}^{-1} = \mathrm{Int}^{\mathrm{R}}(E,D)$, there are rational functions $f_1,\dots,f_r$ in $\mathfrak{B}$ and $g_1,\dots,g_r$ in $\mathfrak{B}^{-1}$ such that $f_1g_1 + \cdots + f_rg_r = 1$. If $x$ is not a pole of any of these functions, we can thus write $f_1(x)g_1(x) + \cdots + f_r(x)g_r(x) = 1$. Since, $f_i(x) \in \mathfrak{B}(x) = \mathfrak{A}(x)$, there are functions $h_i \in \mathfrak{A}$ such that $h_i(x) = f_i(x)$. Therefore, $\mathfrak{J}(x) = D$ for all $x \in E$, except perhaps for finitely many elements (the poles of a finite family of rational functions). However, since $\mathfrak{J}$ is finitely generated, if $\mathfrak{J}(x)$ were contained in some maximal ideal $\mathfrak{m}$ of $D$, then so would $\mathfrak{J}(x+a)$ for each $a$ with sufficiently high value in $D_{\mathfrak{m}}$. So in fact, $\mathfrak{J}(x) = D$ for all $x \in E$. But then it results from the Skolem property [Lemma 4.1] that $\mathfrak{J} = \mathfrak{A}\mathfrak{B}^{-1} = \mathrm{Int}^{\mathrm{R}}(E,D)$. In conclusion $\mathfrak{A} = \mathfrak{B}$. □

**Remark 4.3.** (i) If $\mathrm{Int}^{\mathrm{R}}(K,D)$ is a Bézout domain, there is a direct and easy proof that it satisfies the strong Skolem property (essentially the same as that given for $\mathrm{Int}^{\mathrm{R}}(D)$ in [11, Proposition 3.8]): let $\mathfrak{A}$ and $\mathfrak{B}$ be two finitely generated ideals such that $\mathfrak{A}(x) = \mathfrak{B}(x)$, for all $x \in K$. By hypothesis, $\mathfrak{A} = (\phi)$ and $\mathfrak{B} = (\psi)$ (both ideals are principal), thus, $\rho = (\phi/\psi)$ takes only unit values on $K$ except perhaps for those elements such that $\phi(x) = (\psi(x) = 0)$. This exceptional set is necessarily finite. However, as in the previous proof, it is easy to see that if $\rho(x)$ were contained in some maximal ideal $\mathfrak{m}$ of $D$ then so would $\rho(x+a)$ for each $a$ with sufficiently high value in $D_{\mathfrak{m}}$. Hence, $\rho$

actually takes unit values on all of $K$. Therefore, it is a unit in $\mathrm{Int}^{\mathrm{R}}(K,D)$ and finally $\mathfrak{A} = \mathfrak{B}$.

(ii) For the strong Skolem property, there must be some restrictions on $E$. Suppose, for example that $E$ is a finite set and let $f(X) = \prod_{d \in E}(X-d)$. Then $f$ is identically 0 on $E$ and so is $f^2$, while $f$ and $f^2$ clearly do not generate the same ideal of $\mathrm{Int}^{\mathrm{R}}(E,D)$. For another example (where $E$ may be infinite), let $E = \{0\} \cup U(D)$ (where $U(D)$ is the set of units of $D$). Then $f(X) = X$ takes unit values on $E$, except at 0 where it vanishes, and so does $f^2$. However again, $f$ and $f^2$ do not generate the same ideal of $\mathrm{Int}^{\mathrm{R}}(E,D)$.

The hypothesis of Lemma 4.1 is satisfied if $D$ is a monic Prüfer domain [Theorem 3.2]. In fact, without assuming $D$ to be a Prüfer domain, the existence of a monic unit-valued polynomial is enough to ensure at least the Skolem property.

**Proposition 4.4.** *Let $D$ be a domain such that there exists a unit-valued monic polynomial $f \in D[X]$ and let $E$ be a subset of $K$. Then $\mathrm{Int}^{\mathrm{R}}(E,D)$ has the Skolem property.*

**Proof.** Let $\mathfrak{A} = (\phi_1, \phi_2, \ldots, \phi_r)$ be a finitely generated ideal of $\mathrm{Int}^{\mathrm{R}}(E,D)$ such that $\mathfrak{A}(x) = D$ for all $x \in E$. Write

$$f(X) = a_0 + a_1 X + \cdots + X^n,$$

then define $\psi_1 = \phi_1$ and, for $i > 1$,

$$\psi_i = (\phi_i) \bigstar \psi_{i-1} = a_0\phi_i^n + a_1\phi_i^{n-1}\psi_{i-1} + \cdots + \psi_{i-1}^n = \phi_i^n f\left(\frac{\psi_{i-1}}{\phi_i}\right).$$

By induction, $\psi_i \in \mathfrak{A}$ for each $i$, and in particular $\psi_r \in \mathfrak{A}$. On the other hand, note that for $\alpha$ and $\beta$ in $D$,

$$\alpha \bigstar \beta = a_0\alpha^n + a_1\alpha^{n-1}\beta + \cdots + \beta^n = \alpha^n f\left(\frac{\beta}{\alpha}\right)$$

is such that, if $\alpha$ or $\beta$ does not belong to some maximal ideal $\mathfrak{m}$ of $D$, then neither does $\alpha \bigstar \beta$ belong to this maximal ideal. Therefore, since, for each maximal ideal $\mathfrak{m}$ of $D$ and each $x \in E$ we can find $\phi_i$ such that $\phi_i(x)$ is not in $\mathfrak{m}$, it follows, as in Corollaries 1.13 and 1.15 of [10] that $\psi_r(x)$ is a unit of $D$, for all $x \in E$. This implies that $\psi_r$ is a unit in $\mathrm{Int}^{\mathrm{R}}(E,D)$. In conclusion $\mathfrak{A} = \mathrm{Int}^{\mathrm{R}}(E,D)$. □

## 5. Prime ideals: Generalities

Assuming that $\mathrm{Int}^{\mathrm{R}}(K,D)$ is a Prüfer domain with $K(X)$ as quotient field (for example, when $D$ is a monic or singular Prüfer domain), we may focus much of our attention on the prime ideals of $\mathrm{Int}^{\mathrm{R}}(K,D)$, since $\mathrm{Int}^{\mathrm{R}}(E,D)$ is then an overring of this Prüfer domain. At times, it will also be useful to specifically focus on the prime ideals of $\text{Int}^{\text{R}}(D)$. With this in mind, but first without any hypothesis on the domain $D$, we first consider the prime ideals above $(0)$.

**Lemma 5.1.** *Let $D$ be a domain, with quotient field $K$ and $g$ be a nonconstant polynomial in $K[X]$. Then the following assertions are equivalent:*

(1) *there exists a nonzero polynomial $h \in K[X]$, relatively prime to $g$ in $K[X]$ such that $h/g \in \text{Int}^{\text{R}}(D)$,*

(2) *there exists a nonzero constant $a \in D$ such that $a/g \in \text{Int}^{\text{R}}(D)$.*

**Proof.** If $h$ and $g$ are relatively prime, there exists two polynomials $h_1$ and $g_1$ and a nonzero constant $a$ such that $h_1 h + g_1 g = a$. We may as well suppose that these polynomials have their coefficients in $D$, hence that they are integer-valued (we may also suppose that $a \in D$). If $h/g \in \text{Int}^{\text{R}}(D)$, then $h_1(h/g) + g_1(g/g) = a/g$ also belongs to $\text{Int}^{\text{R}}(D)$. Therfore (i) implies (ii). The converse is obvious. □

Under these equivalent conditions, $g$ is invertible in the localization $S^{-1}\,\text{Int}^{\text{R}}(D)$, where $S$ is the complement of $(0)$ in $D$. In other words, $S^{-1}\,\text{Int}^{\text{R}}(D) = T^{-1}K[X]$, where $T$ is the multiplicative set formed by these polynomials. The following result is then immediate.

**Proposition 5.2.** *The nonzero prime ideals of $\text{Int}^{\text{R}}(D)$ above $(0)$ are in one-to-one correspondence with the monic irreducible polynomials of $K[X]$ which never appear as the denominator of an irreducible integer-valued rational function. To the irreducible polynomial $q$ corresponds the prime ideal*

$$\mathfrak{P}_q = qK[X] \cap \text{Int}^{\text{R}}(D).$$

**Remark 5.3.** (i) Polynomials which do not appear as the denominator of any irreducible integer-valued rational function certainly exist: for any $d \in D$, consider for instance, the polynomial $q(X) = X - d$. Accordingly, there is always a chain $(0) \subset \mathfrak{P}_q$ of primes above $(0)$.

(ii) If we restrict ourselves to the case where $D = V$ is a valuation domain (as in the next section), it is clear that $g$ satisfies the equivalent conditions of the previous lemma if and only if $\{v(g(x)) \mid x \in D\}$ is bounded. The nonzero prime ideals of $\text{Int}^{\text{R}}(V)$ above $(0)$ then correspond to the irreducible polynomials of $K[X]$ such that $\{v(g(x)) \mid x \in D\}$ is unbounded.

Considering even a subset $E$ of $K$, we now give very general examples of prime ideals of $\text{Int}^{\text{R}}(E, D)$. We first observe that, for each pair $(\mathfrak{p}, x)$, where $\mathfrak{p}$ is a prime ideal of $D$ and $x \in E$, then

$$\mathfrak{P}_{\mathfrak{p},x} = \{\phi \in \text{Int}^{\text{R}}(E, D) \mid \phi(x) \in \mathfrak{p}\}$$

is a prime ideal of $\mathrm{Int}^{\mathrm{R}}(E,D)$ above $\mathfrak{p}$. We shall say that such a prime ideal is a *pointed ideal* (similar prime ideals occur in the study of integer-valued polynomials, see for instance [6]). We already considered such ideals in the previous section; also, letting $\mathfrak{p} = (0)$, and $E = D$, it is easy to see that the prime ideal $\mathfrak{P}_{(0),x}$ is nothing else than the prime ideal $\mathfrak{P}_{(X-x)}$ of Proposition 5.2.

We obtain more ideals by the consideration of ultrafilters. We first set a definition and a notation:

**Definition 5.4.** Let $D$ be a domain, $E$ be a subset of $K$, and $H$ be a (nonempty) set of pairs $(\mathfrak{p},x)$, where $\mathfrak{p}$ is a prime ideal of $D$ and $x \in E$. For each $\phi \in \mathrm{Int}^{\mathrm{R}}(E,D)$, let

$$B_{(\phi,H)} = \{(\mathfrak{p},x) \in H \mid \phi(x) \in \mathfrak{p}\}.$$

We say that $B_{(\phi,H)}$ is the *characteristic set* of $\phi$ on $H$.

We can make a few easy observations:
– The characteristic set of the function 0 is $H$.
– Let $\phi,\psi \in \mathrm{Int}^{\mathrm{R}}(E,D)$. Then $B_{(\phi,H)} \cap B_{(\psi,H)} \subseteq B_{(\phi+\psi,H)}$.
– Let $\phi,\psi \in \mathrm{Int}^{\mathrm{R}}(E,D)$. Then $B_{(\phi,H)} \cup B_{(\psi,H)} = B_{(\phi\psi,H)}$.

**Proposition 5.5.** *Let $D$ be a domain, $E$ be a subset of $K$, and $U$ be a filter on a (nonempty) set $H$ of pairs $(\mathfrak{p},x)$, where $\mathfrak{p}$ is a prime ideal of $D$ and $x \in E$. Let*

$$\mathfrak{P}_U = \{\phi \in \mathrm{Int}^{\mathrm{R}}(E,D) | B_{(\phi,H)} \in U\}.$$

*Then $\mathfrak{P}_U$ is a proper ideal of* $\mathrm{Int}^{\mathrm{R}}(E,D)$. *If moreover $U$ is an ultrafilter, then $\mathfrak{P}_U$ is a prime ideal.*

**Proof.** First note that $0 \in \mathfrak{P}_U$ so $\mathfrak{P}_U$ is not empty. Let $\phi,\psi \in \mathfrak{P}_U$. Then $B_{(\phi+\psi,H)}$ belongs to $U$ since it contains $B_{(\phi,H)} \cap B_{(\psi,H)}$. Similarly, if $\phi \in \mathfrak{P}_U$ and $\rho \in \mathrm{Int}^{\mathrm{R}}(E,D)$, then $B_{(\rho\phi,H)}$ contains $B_{(\phi,H)}$, thus $\rho\phi \in \mathfrak{P}_U$. In conclusion, $\mathfrak{P}_U$ is an ideal. It is a proper ideal since the empty set does not belong to $U$ and therefore, the characteristic set of a unit does not belong to $U$.

Finally, suppose that $\phi\psi \in \mathfrak{P}_U$. Then $B_{(\phi,H)} \cup B_{(\psi,H)} = B_{(\phi\psi,H)} \in U$. If $U$ is an ultrafilter, either $B_{(\phi,H)}$ or $B_{(\psi,H)}$ must belong to $U$, hence either $\phi$ or $\psi$ must belong to $\mathfrak{P}_U$. Therefore, $\mathfrak{P}_U$ is a prime ideal. $\square$

**Remark 5.6.** (i) In particular, if we choose a pair $(\mathfrak{p},x)$ in the set $\Omega$ of all such pairs and let $U$ be the principal ultrafilter consisting of all subsets of $\Omega$ containing $(\mathfrak{p},x)$, then $\mathfrak{P}_U$ is the pointed ideal $\mathfrak{P}_{\mathfrak{p},x}$.

(ii) More generally, choosing a prime $\mathfrak{p}$ and letting $H_{\mathfrak{p}}$ be the set of pairs $(\mathfrak{p},x)$, where $x \in E$, then $\mathfrak{P}_U$ is a prime ideal of $\mathrm{Int}^{\mathrm{R}}(E,D)$ above $\mathfrak{p}$. In this case it is simpler to consider $U$ as an ultrafilter on $E$. If, in particular, we consider the ring $\mathrm{Int}^{\mathrm{R}}(V)$ of integral-value rational functions on a rank-one discrete valuation domain $V$ with finite residue field, ultrafilters on $V$ correspond to the completion $\widehat{V}$ of $V$ and we, thus, recover the prime ideals of $\mathrm{Int}^{\mathrm{R}}(V)$ above the maximal ideal $\mathfrak{m}$ of $V$ as in [3].

It is not clear whether all prime ideals of $\mathrm{Int}^{\mathrm{R}}(E,D)$ are ultrafilter ideals, nevertheless, if we suppose that $\mathrm{Int}^{\mathrm{R}}(E,D)$ is a Prüfer domain with torsion Picard group, ultrafilters on the set $\Omega$ of all pairs $(\mathfrak{p},x)$ yield all maximal ideals of $\mathrm{Int}^{\mathrm{R}}(E,D)$.

**Proposition 5.7.** *Let $D$ be a domain and $E$ be a subset of $K$ such that* $\mathrm{Int}^{\mathrm{R}}(E,D)$ *is a Prüfer domain with torsion Picard group. Then, for each proper ideal $\mathfrak{A}$ of* $\mathrm{Int}^{\mathrm{R}}(E,D)$, *there exists an ultrafilter $U$ on the set $\Omega$ of all ordered pairs $(\mathfrak{p},x)$, where $\mathfrak{p}$ is a prime ideal of $D$ and $x \in E$, such that $\mathfrak{A} \subseteq \mathfrak{P}_U$.*

**Proof.** Let $U_{\mathfrak{A}}$ be the collection of charateristic sets $B_{\phi,\Omega}$, where $\phi \in \mathfrak{A}$. Note that $B_{\phi,\Omega}$ is empty if and only if $\phi$ is a unit. Therefore, $U_{\mathfrak{A}}$ does not contain the empty set. We next claim that $U_{\mathfrak{A}}$ is closed under finite intersection. Let $\phi_1, \phi_2$ be elements of $\mathfrak{A}$. They generate an ideal $\mathfrak{B}$ and, by hypothesis, there exists a positive integer $e$ such that $\mathfrak{B}^e$ is principal: $\mathfrak{B}^e = (\psi)$. Clearly, $B_{\psi,\Omega} = B_{\phi_1,\Omega} \cap (B_{\phi_2,\Omega})$. Then it is well known that $U_{\mathfrak{A}}$ can be extended to an ultrafilter $U$ on $\Omega$ and hence, $\mathfrak{A} \subseteq \mathfrak{P}_U$. □

## 6. Prime ideals: Valuation domain

From now on we focus our attention on the special case, where $D = V$ is a valuation domain such that $\mathrm{Int}^{\mathrm{R}}(E,V)$ is a Prüfer domain with torsion Picard group, for example, when its maximal ideal is principal or its residue field is not algebraically closed. We let $v$ be a valuation corresponding to $V$ and $\mathfrak{m}$ be the maximal ideal of $V$.

To study the prime ideals above a prime ideal $\mathfrak{p}$, we consider ultrafilters on the set of pairs $(\mathfrak{p},x)$, where $x \in E$, in other words ultrafilters on $E$. For each $\phi \in \mathrm{Int}^{\mathrm{R}}(E,V)$, as in Definition 5.4, we then define the $\mathfrak{p}$-characteristic set of $\phi$ as

$$B_{\phi,\mathfrak{p}} = \{x \in E \mid \phi(x) \in \mathfrak{p}\},$$

and to an ultrafilter $U$ we associate the prime ideal

$$\mathfrak{P}_{\mathfrak{p},U} = \{\phi \in \mathrm{Int}^{\mathrm{R}}(E,V) \mid B_{(\phi,\mathfrak{p})} \in U\}.$$

It is clear that the $\mathfrak{m}$-characterisitic set $B_{\phi,\mathfrak{m}}$ of a function $\phi$ is empty if and only if $\phi$ is a unit. With the same proof as in Proposition 5.7, we then have the following:

**Proposition 6.1.** *Let $V$ be a valuation domain and $E$ be a subset of $K$ such that* $\mathrm{Int}^{\mathrm{R}}(E,V)$ *is a Prüfer domain with torsion Picard group. Then each maximal ideal of* $\mathrm{Int}^{\mathrm{R}}(E,V)$ *is of the form*

$$\mathfrak{M}_{\mathfrak{m},U} = \{\phi \in \mathrm{Int}^{\mathrm{R}}(E,V) \mid B_{\phi,\mathfrak{m}} \in U\}$$

*for some ultrafilter $U$ on $E$. In particular, if $\mathfrak{M}$ is a maximal ideal of* $\mathrm{Int}^{\mathrm{R}}(E,V)$ *then $\mathfrak{M} \cap V = \mathfrak{m}$.*

If $\mathfrak{M} = \mathfrak{M}_{\mathfrak{m},U}$ is a maximal ideal of the Prüfer domain $\mathrm{Int}^{\mathrm{R}}(E,V)$, there is a single chain of prime ideals contained in $\mathfrak{M}$. If $\mathfrak{p}$ is contained in $\mathfrak{m}$, then $\mathfrak{M}_{\mathfrak{m},U}$, contains the prime ideal $\mathfrak{P}_{\mathfrak{p},\mathfrak{U}}$ above $\mathfrak{p}$, corresponding to the same filter on $E$. Hence, the prime $\mathfrak{P}_{\mathfrak{p},U}$ appears in the chain of primes contained in $\mathfrak{M}$. In particular, if $V$ is a valuation domain of finite dimension $r$, we know that $\mathrm{Int}^{\mathrm{R}}(E,V)$ is of dimension at most $r+1$ (since $V \subseteq \mathrm{Int}^{\mathrm{R}}(E,V) \subseteq K(X)$ and since $V$ is a Jaffard domain [1, Lemme 1.1]). Since the prime ideals of the form $\mathfrak{P}_{\mathfrak{p},U}$ form a chain of length $r$, we have the following.

**Corollary 6.2.** *Let $V$ be a valuation domain of finite dimension $r$ and $E$ be a subset of $K$ such that $\mathrm{Int}^{\mathrm{R}}(E,V)$ is a Prüfer domain with torsion Picard group. Then,*

(1) *each saturated chain of primes in $\mathrm{Int}^{\mathrm{R}}(E,V)$ is of length $r$ or $r+1$,*

(2) *for each maximal ideal $\mathfrak{M}$ of $\mathrm{Int}^{\mathrm{R}}(E,V)$ and each prime ideal $\mathfrak{p}$ of $V$ there is a prime ideal of $\mathrm{Int}^{\mathrm{R}}(E,V)$ above $\mathfrak{p}$ contained in $\mathfrak{M}$,*

(3) *for each prime ideal $\mathfrak{p}$ of $V$, each chain of primes in $\mathrm{Int}^{\mathrm{R}}(E,V)$ above $\mathfrak{p}$ has length at most one.*

We have seen in the previous section that there is always a chain of length one above $(0)$. For the prime ideals above $\mathfrak{m}$, a significant difference will appear between the two classes of valuation domains we have considered: in one case, all these prime ideals are maximal, in the other, there may be a chain of length one above $\mathfrak{m}$. We first consider the case where $\mathfrak{m}$ is principal, generated by $t$.

**Theorem 6.3.** *Let $V$ be a valuation domain such that the maximal ideal $\mathfrak{m}$ is principal, generated by $t$. Let $E$ be a subset of $K$ and $\mathfrak{M}$ be a prime ideal of $\mathrm{Int}^{\mathrm{R}}(E,V)$ which lies over $\mathfrak{m}$. Then $\mathfrak{M}$ is maximal.*

**Proof.** Choose $\psi \in \mathrm{Int}^{\mathrm{R}}(E,V)$ but $\psi \notin \mathfrak{M}$. Then let $\phi = \psi/(t+\psi^2)$. It is easy to see that $x/(t+x^2) \in V$ for all $x \in K$. Hence, $\phi \in \mathrm{Int}^{\mathrm{R}}(K,V)$ and a fortiori $\phi \in \mathrm{Int}^{\mathrm{R}}(E,V)$. Consider $\psi(1-\psi\phi) = \psi t/(t+\psi^2) = t\phi$. Since $\mathfrak{M}$ contains $t$, it contains the product $\psi(1-\psi\phi)$. Since $\mathfrak{M}$ does not contain $\psi$ it contains $(1-\psi\phi)$. Hence $\psi$ is invertible modulo $\mathfrak{M}$. □

Since $\mathrm{Int}^{\mathrm{R}}(E,V)$ is a localization of $\mathrm{Int}^{\mathrm{R}}(K,V)$, we could restrict ourselves to the study of $\mathrm{Int}^{\mathrm{R}}(K,V)$. In fact, using our ultrafilter characterization of the maximal ideals, we may even restrict ourselves to $\mathrm{Int}^{\mathrm{R}}(V)$. Note that the field isomorphism $\alpha$ of $K(X)$ defined by $\alpha(\phi(X)) = \phi(1/X)$ fixes $\mathrm{Int}^{\mathrm{R}}(K,V)$, hence $\alpha$ permutes the prime ideals of $\mathrm{Int}^{\mathrm{R}}(K,V)$. From Proposition 2.1, a maximal ideal $\mathfrak{M}$ of $\mathrm{Int}^{\mathrm{R}}(K,V)$ survives in $\mathrm{Int}^{\mathrm{R}}(V)$ if and only if, for each $\phi \in \mathfrak{M}$, there is $x \in V$ such that $\phi(x) \in \mathfrak{m}$. Since the corresponding ultrafilter $U$ either contains $V$ or its complement, the maximal ideals which lift in $\mathrm{Int}^{\mathrm{R}}(V)$ are then precisely those corresponding to an ultrafilter which contains $V$. Therefore, if $\mathfrak{M}$ is a maximal ideal of $\mathrm{Int}^{\mathrm{R}}(K,V)$ either it survives in $\mathrm{Int}^{\mathrm{R}}(V)$ or its image via the isomorphism $\alpha$ does. The same is then true of each prime ideal, in particular, we may conlude that $\mathrm{Int}^{\mathrm{R}}(K,V)$ and $\mathrm{Int}^{\mathrm{R}}(V)$ have the same Krull dimension (possibly infinite). From now on, we shall thus restrict ourselves to the study of $\mathrm{Int}^{\mathrm{R}}(V)$.

We then consider another type of prime ideal. First, we set some notations: if $\mathfrak{p}$ is a prime ideal of $V$, we denote by $v_{\mathfrak{p}}$ the valuation of $K$ corresponding to the localization $V_{\mathfrak{p}}$, by $v_{\mathfrak{p}}^*$ the valuation of $K(X)$ defined on $K[X]$ by

$$v_{\mathfrak{p}}^*(a_0 + a_1X + \cdots + a_nX^n) = \inf_0^n v_{\mathfrak{p}}(a_i),$$

and lastly by $V_{\mathfrak{p}}^*$ the corresponding valuation domain. With these notations, we have the following, which generalizes [3, Lemme 3.5].

**Proposition 6.4.** *Let $V$ be a valuation domain and $\mathfrak{p}$ be a prime ideal such that $V/\mathfrak{p}$ is infinite or $\mathfrak{p}V_{\mathfrak{p}}$ is not principal. Then $\mathrm{Int}^{\mathrm{R}}(V)$ is contained in $V_{\mathfrak{p}}^*$.*

**Proof.** Let $\phi \in \mathrm{Int}^{\mathrm{R}}(V)$ and suppose that $\phi \neq 0$. We can write $\phi = d(g/h)$ with $d \in K$ and $g$ and $h$ in $V[X]$, relatively prime over $K[X]$ and such that $v_{\mathfrak{p}}^*(g) = v_{\mathfrak{p}}^*(h) = 0$. Then $v_{\mathfrak{p}}^*(\phi) = v_{\mathfrak{p}}(d)$ and we need to show that $v_{\mathfrak{p}}(d) \geq 0$. For each $a \in V$, since $\phi(a) \in V$, we have

$$v_{\mathfrak{p}}(\phi(a)) = v_{\mathfrak{p}}(d) + v_{\mathfrak{p}}(g(a)) - v_{\mathfrak{p}}(h(a)) \geq 0.$$

A fortiori $v_{\mathfrak{p}}(dg(a)) \geq 0$ for each $a \in V$, in other words, $dg$ is an element of the ring $\mathrm{Int}(V, V_{\mathfrak{p}})$ of polynomials taking $V$ into $V_{\mathfrak{p}}$. It is known that $\mathrm{Int}(V, V_{\mathfrak{p}}) = \mathrm{Int}(V_{\mathfrak{p}})$, the ring of integer-valued polynomials on $V_{\mathfrak{p}}$ [4, Proposition 5, Corollary 1] and that $\mathrm{Int}(V_{\mathfrak{p}}) = V_{\mathfrak{p}}[X]$ provided that the maximal ideal of $V_{\mathfrak{p}}$ is not principal [5, Proposition 1.2] or that its residue field is infinite [4, Proposition 5, Corollary 2]. This clearly completes the proof. □

Under the conditions of this proposition, the maximal ideal of the valuation ring $V_{\mathfrak{p}}^*$ contracts in $\mathrm{Int}^{\mathrm{R}}(V)$ to the prime ideal

$$\mathfrak{P}_{\mathfrak{p}}^* = \{\phi \in \mathrm{Int}^{\mathrm{R}}(V) \mid v_{\mathfrak{p}}^*(\phi) \geq 0\}.$$

This prime ideal is obviously above the prime ideal $\mathfrak{p}$ of $V$. In particular, if the maximal ideal $\mathfrak{m}$ of $V$ is not principal or such that $V/\mathfrak{m}$ is infinite, writing simply $v^*$ for $v_{\mathfrak{m}}^*$ and $V^*$ for $V_{\mathfrak{m}}^*$, then $\mathrm{Int}^{\mathrm{R}}(V)$ is contained in $V^*$ and the maximal ideal of $V^*$ contracts to a prime ideal above $\mathfrak{m}$ of the form

$$\mathfrak{M}^* = \{\phi \in \mathrm{Int}^{\mathrm{R}}(V) \mid v^*(\phi) \geq 0\}.$$

**Remark 6.5.** We do not suppose here that $\mathrm{Int}^{\mathrm{R}}(V)$ is a Prüfer domain. For instance, we have seen that, if the quotient field of $V$ is algebraically closed, then $\mathrm{Int}^{\mathrm{R}}(V) = S^{-1}V[X]$, where $S = \{rX + 1 \mid r \in \mathfrak{m}\}$ [Proposition 2.4] and $\mathrm{Int}^{\mathrm{R}}(V)$ is not a Prüfer domain [Remark 3.6 (v)]. Yet, it remains true that $\mathrm{Int}^{\mathrm{R}}(V)$ is contained in $V^*$.

In Theorem 6.3, we have shown that if $\mathfrak{m}$ is principal, then each prime ideal of $\mathrm{Int}^{\mathrm{R}}(E, V)$ which lies over $\mathfrak{m}$ is maximal. We show now that if $\mathfrak{m}$ is not principal, then the prime ideal $\mathfrak{M}^*$, which is above $\mathfrak{m}$ is not maximal (adding the hypothesis that $V/\mathfrak{m}$ is not algebraically closed, then $\mathrm{Int}^{\mathrm{R}}(V)$ would nevertheless be a Prüfer domain, and if moreover the dimension of $V$ is supposed to be finite, then each prime ideal strictly containing $\mathfrak{M}^*$ would be maximal [Corollary 6.2]).

**Theorem 6.6.** *Let $V$ be a valuation domain such that $\mathfrak{m}$, the maximal ideal of $V$, is not principal. Then the prime ideal $\mathfrak{M}^*$ of $\mathrm{Int}^{\mathrm{R}}(V)$ is not maximal.*

**Proof.** Let $U$ be a nonprincipal ultrafilter on $V$ containing, for each $\varepsilon$ in the value group of $V$, the subset $A_\varepsilon = \{x \in \mathfrak{m} \mid v(x) < \varepsilon\}$. Then let $\mathfrak{M}_U$ be the corresponding prime ideal of $\mathrm{Int}^{\mathrm{R}}(V)$ lying over $\mathfrak{m}$. Let $\phi \in \mathfrak{M}^*$. Recall that we can write $\phi = d(g/h)$ where $v^*(g) = v^*(h) = 0$ and $d \in m$. It is easy to see that if $v(x)$ is small enough, then $v(h(x)) < v(d)$, hence $v(d(g(x)/h(x))) \geq 0$. Hence $\phi \in \mathfrak{M}_U$. Therefore, $\mathfrak{M}^* \subseteq \mathfrak{M}_U$. It is also easy to see that $X \in \mathfrak{M}_U$ and that $X \notin \mathfrak{M}^*$. □

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